Hierarchical Diffusion Curves for Accurate Automatic Image Vectorization

1 Integration over a Rectangle

Motivated by Sun et al. [2012], in order to more efficiently reconstruct anti-aliased results, we have derived closed-form analytic solutions to the image reconstruction integral $u(\mathbf{x})$ (in Eq. (2) of the paper) over a rectangular region $\mathbf{R}$, as opposed to simply evaluating it e.g. at a pixel center point $\mathbf{x}$:

$$\phi(\mathbf{R}) = \iint\limits_{\mathbf{R}} u(\mathbf{x})\ \mathrm{d}\mathbf{x}. \tag{1}$$

The integral in Eq. (1) of $u(\mathbf{x})$ over a rectangular region $\mathbf{R}$ can be expressed in terms of integrations of the Green's function kernels $G^L(\mathbf{x},\mathbf{x}')$, $\partial G^L(\mathbf{x},\mathbf{x}')/\partial\mathbf{n}(\mathbf{x}')$, $G^B(\mathbf{x},\mathbf{x}')$, and $\partial G^B(\mathbf{x},\mathbf{x}')/\partial\mathbf{n}(\mathbf{x}')$ over $\mathbf{R} = \{x \in (x_0, x_1), y \in (y_0, y_1)\}$:

$$\begin{aligned}\phi(\mathbf{R}) = &\iint\limits_{\mathbf{R}} \left(\oint\limits_{\partial\mathbf{D}} \left(\frac{\partial u(\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')} G^L(\mathbf{x},\mathbf{x}') - u(\mathbf{x}')\, G^L_n(\mathbf{x},\mathbf{x}') \right) \mathrm{d}\mathbf{x}' \right) \mathrm{d}\mathbf{x} \\ &+ \iint\limits_{\mathbf{R}} \left(\oint\limits_{\partial\mathbf{D}} \left(\frac{\partial v(\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')} G^B(\mathbf{x},\mathbf{x}') - v(\mathbf{x}')\, G^B_n(\mathbf{x},\mathbf{x}') \right) \mathrm{d}\mathbf{x}' \right) \mathrm{d}\mathbf{x}\end{aligned} \tag{2}$$

where $G^L_n(\mathbf{x},\mathbf{x}') = \frac{\partial G^L(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')}$, $G^B_n(\mathbf{x},\mathbf{x}') = \frac{\partial G^B(\mathbf{x},\mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')}$.

As derived in Sun et al. [2012], closed-form integrals for $F_{G^L}(\mathbf{R},\mathbf{x}') = \iint_{\mathbf{R}} G^L(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x}$ and $F_{G^L_n}(\mathbf{R},\mathbf{x}') = \iint_{\mathbf{R}} G^L_n(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x}$ exist for this Green's function over a rectangular region $\mathbf{R}$.

We derive new closed-form integrals $F_{G^B}(\mathbf{R},\mathbf{x}') = \iint_{\mathbf{R}} G^B(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x}$ for the bilaplacian term G^B:

$$F_{G^B}(\mathbf{R},\mathbf{x}') = \sum_{i,j\in\{0,1\}} (-1)^{i+j}\, H_{G^B}(\hat{x},\hat{y}) \tag{3}$$

and $F_{G^B_n}(\mathbf{R},\mathbf{x}') = \iint_{\mathbf{R}} G^B_n(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x}$ for the bilaplacian normal term G^B_n:

$$F_{G^B_n}(\mathbf{R},\mathbf{x}') = \sum_{i,j\in\{0,1\}} (-1)^{i+j}\, H_{G^B_n}(\hat{x},\hat{y},n_x,n_y) \tag{4}$$

where $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y')$, $\mathbf{n}(\mathbf{x}') = (n_x, n_y)$, and $(\hat{x},\hat{y}) = \mathbf{x} - \mathbf{x}'$. Here, we define $H_{G^B}(\hat{x},\hat{y})$ in Eq. (3) as

$$\begin{aligned}H_{G^B}(\hat{x},\hat{y}) &= \iint G^B(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x} \\ &= \frac{1}{8\pi}\int\left(\int\left(\hat{x}^2+\hat{y}^2\right)\left(\ln\left(\frac{1}{\sqrt{\hat{x}^2+\hat{y}^2}}\right)+1\right)\mathrm{d}\hat{x}\right)\mathrm{d}\hat{y} \\ &= \frac{1}{144\pi}\int\left(8\hat{x}^3+30\hat{x}\hat{y}^2-12\hat{y}^3\,\mathrm{atan}\left(\frac{\hat{x}}{\hat{y}}\right)\right)\mathrm{d}\hat{y} \\ &\quad - \frac{1}{144\pi}\int\left(3\left(\hat{x}^3+3\hat{x}\hat{y}^2\right)\ln\left(\hat{x}^2+\hat{y}^2\right)\right)\mathrm{d}\hat{y} \\ &= \frac{1}{48\pi}\left(\hat{x}^4-\hat{y}^4\right)\mathrm{atan}\left(\frac{\hat{x}}{\hat{y}}\right) \\ &\quad + \frac{1}{144\pi}\hat{x}\hat{y}\left(\hat{x}^2+\hat{y}^2\right)\left(11-3\ln\left(\hat{x}^2+\hat{y}^2\right)\right)\end{aligned} \tag{5}$$

Similarly, $H_{G^B_n}(\hat{x},\hat{y},n_x,n_y)$ in Eq. (4) is defined as:

$$\begin{aligned}H_{G^B_n}(\hat{x},\hat{y},n_x,n_y) &= \iint G^B_n(\mathbf{x},\mathbf{x}')\ \mathrm{d}\mathbf{x} \\ &= -\frac{1}{8\pi}\int\left(\int\left(\hat{x}n_x+\hat{y}n_y\right)\left(-1+\ln\left(\hat{x}^2+\hat{y}^2\right)\right)\mathrm{d}\hat{x}\right)\mathrm{d}\hat{y} \\ &= \frac{1}{16\pi}\int\left(2\hat{x}\left(\hat{x}n_x+3\hat{y}n_y\right)+4\hat{y}^2n_y\,\mathrm{atan}\left(\frac{\hat{y}}{\hat{x}}\right)\right)\mathrm{d}\hat{y} \\ &\quad - \frac{1}{16\pi}\int\left(\left(2\hat{x}\hat{y}n_y+\left(\hat{x}^2+\hat{y}^2\right)n_x\right)\ln\left(\hat{x}^2+\hat{y}^2\right)\right)\mathrm{d}\hat{y} \\ &= \frac{1}{48\pi}\left(10\hat{x}\hat{y}\left(\hat{x}n_x+\hat{y}n_y\right)-4\hat{y}^3n_y\,\mathrm{atan}\left(\frac{\hat{x}}{\hat{y}}\right)-4\hat{x}^3n_x\,\mathrm{atan}\left(\frac{\hat{y}}{\hat{x}}\right)\right) \\ &\quad - \frac{1}{48\pi}\left(\hat{x}^3n_y+3\hat{x}^2\hat{y}n_x+3\hat{x}\hat{y}^2n_y+\hat{y}^3n_x\right)\ln\left(\hat{x}^2+\hat{y}^2\right)\end{aligned} \tag{6}$$

References

SUN, X., XIE, G., DONG, Y., LIN, S., XU, W., WANG, W., TONG, X., AND GUO, B. 2012. Diffusion curve textures for resolution independent texture mapping. *ACM Trans. Graph. 31* (July), 74:1–74:9.